ADAPTIVE FINITE ELEMENT TECHNIQUE FOR VISCOUS FLOW ANALYSIS

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ABSTRACT

A finite element method for analysis of viscous incompressible flow is presented. The finite element formulation and the computational procedure are described. The corresponding computer program that can be executed on standard personal computers has been developed. The method is combined with adaptive meshing technique to increase the solution accuracy, and at the same time, to minimise the computational time and computer memory required. The efficiency of the combined method is evaluated by examples of flow past a cylinder, flow in a sudden enlargement duct, and flow through an orifice.

1. INTRODUCTION

The use of the finite element method to solve fluid flow problems has increased its popularity. This is due to the fact that the governing differential equations for general flow problems consist of several coupled equations which are inherently nonlinear. Together with complex flow geometries and general boundary conditions that increase the difficulty of the problem, the finite element method is more attractive comparing to other numerical methods¹⁻³. Despite its capability and popularity as mentioned, accurate finite element solution still requires a vast amount of computer time and data storage⁴⁻⁵. One way to minimise the amount of computer time and data storage used is to employ adaptive mesh-generation techniques. The techniques may be classified into two major categories - refinement/derefinement and remeshing. The first category, the adaptive refinement/derefinement technique, can be further classified into three subcategories: (a) the *h* method, (b) the *p* method, and (c) the *r* method. In the *h* method, the elements in the initial mesh are refined into smaller elements or derefined into larger elements⁶. The *p* method maintains the geometry of the elements of the initial mesh but increases (or decreases) the order of the polynomials used for the element interpolation functions⁷. The *r* method keeps the number of elements and their connectivities the same but relocates the nodes⁸.

In this paper, the finite element method for viscous incompressible flows⁹ is combined with the adaptive meshing technique^{10,11} to improve the finite element analysis solution

accuracy and reduce the computational time as well as the computer memory. The adaptive meshing technique will place small elements in the region of large change in the solution gradients to increase solution accuracy, and at the same time, will place larger elements in the other regions to reduce the computational time and computer memory.

The paper starts by explaining the finite element formulation and the corresponding solution procedure that leads to the development of a computer program. The basic idea behind the adaptive meshing technique is then described. Finally, the efficiency of the combined method is evaluated by examples of flow past a cylinder, flow in a sudden enlargement duct, and flow through an orifice.

2. THEORETICAL FORMULATION AND SOLUTION PROCEDURE

2.1 Governing Equations

The fundamental laws used to solve fluid motion in a general form¹² are: (a) conservation of mass or continuity equation, (b) conservation of momentum, and (c) conservation of energy, which constitute a set of coupled, nonlinear, partial differential equations.

For low-speed incompressible flow, the fluid density and the flow temperature are assumed to be constant, and only the continuity equation and the momentum equations are needed for the analysis. These differential equations for the two-dimensional steady-state flow are,

$$uu_{,x} + vu_{,y} - \sigma_{x,x} - \tau_{xy,y} = 0 \quad (1a)$$

$$uu_{,x} + vu_{,y} - \sigma_{x,x} - \tau_{xy,y} = 0 \quad (1b)$$

$$uv_{,x} + vv_{,y} - \tau_{xy,x} - \sigma_{y,y} = 0 \quad (1c)$$

where u and v are the velocity components in the x and y direction, respectively. The stress components σ_x, σ_y and τ_{xy} are defined by

$$\sigma_x = \frac{-p}{\rho} + 2vu_{,x} \quad (2a)$$

$$\sigma_y = \frac{-p}{\rho} + 2vv_{,y} \quad (2b)$$

$$\tau_{xy} = \nu(u_{,y} + v_{,x}) \quad (2c)$$

and where p is the pressure, ν is the kinetic viscosity which can be written in form of the viscosity, μ , and density, ρ , as,

$$\nu = \frac{\mu}{\rho} \quad (3)$$

The differential equations, 1a – c, are to be solved with appropriate boundary conditions which are either specifying velocity components along edge S_1 ,

$$u(x,y) = u_1(x,y) \quad (4a)$$

$$v(x,y) = v_1(x,y) \quad (4b)$$

or surface tractions along edge S_2 ,

$$T_x = \sigma_x l + \tau_{xy} m \quad (5a)$$

$$T_y = \tau_{xy} l + \sigma_y m \quad (5b)$$

and l and m are the direction cosines of the unit vector normal to the boundary edge.

2.2 Finite element formulation

The basic unknowns for the two-dimensional viscous incompressible flow problem corresponding to the continuity Equation 1a and the two momentum Equations 1b–c are the velocity components u , v and the pressure p . The six-node triangular element suggested in [4] is used in this study. The element assumes quadratic interpolation for the velocity component distributions and linear interpolation for the pressure distribution according to their highest derivative orders in the differential Equations 1a–c as,

$$u(x,y) = N_\alpha u_\alpha \quad (6a)$$

$$v(x,y) = N_\alpha v_\alpha \quad (6b)$$

$$p(x,y) = H_\lambda p_\lambda \quad (6c)$$

where $\alpha = 1,2,\dots,6$; $\gamma = 1,2,3$; N_α and H_γ are the element interpolation functions for the velocity and pressure, respectively.

To derive the finite element equations, the method of weighted residuals¹³ is applied to the momentum Equations 1b–c and the continuity Equation 1a,

$$\int_A N_i (u u_{,x} + v u_{,y} - \sigma_{x,x} - \tau_{xy,y}) dA = 0 \quad (7a)$$

$$\int_A N_i (u v_{,x} + v v_{,y} - \tau_{xy,x} - \sigma_{y,y}) dA = 0 \quad (7b)$$

$$\int_A H_i (u_{,x} + v_{,y}) dA = 0 \quad (7c)$$

where A is the element area. Gauss' theorem¹³ is then applied to Equations (7a–b) for generating the element boundary integrals. The process gives,

$$\int_A N_i (u_{,x} + v_{,y}) dA + \int_A (N_{i,x} \sigma_x) dA + \int_A (N_{i,y} \tau_{xy}) dA = \int_{S_2} N_i T_x dS \quad (8a)$$

$$\int_A N_i (u_{,y} + v_{,x}) dA + \int_A (N_{i,x} \tau_x) dA + \int_A (N_{i,y} \sigma_y) dA = \int_{S_2} N_i T_y dS \quad (8b)$$

$$\int_A H_i (u_{,x} + v_{,y}) dA = 0 \quad (8c)$$

By substituting the stress components in terms of pressure and velocity gradients from Equations 2a-c, the above Equations 8a-b become,

$$\begin{aligned} \int_A N_i u_{,x} dA + \int_A N_i v_{,y} dA - \int_A N_{i,x} \frac{p}{\rho} dA \\ + \int_A 2v N_{i,x} u_{,x} dA + \int_A v N_{i,y} u_{,y} dA + \int_A v N_{i,y} v_{,x} dA \\ = \int_{S_2} N_i T_x dS \end{aligned} \quad (9a)$$

$$\begin{aligned} \int_A N_i u_{,y} dA + \int_A N_i v_{,x} dA - \int_A N_{i,y} \frac{p}{\rho} dA \\ + \int_A v N_{i,x} u_{,y} dA + \int_A v N_{i,x} v_{,x} dA + \int_A 2v N_{i,y} v_{,y} dA \\ = \int_{S_2} N_i T_y dS \end{aligned} \quad (9b)$$

From the assumed element velocity components, their derivatives are,

$$u_{,x} = N_{\alpha,x} u_\alpha \quad u_{,y} = N_{\alpha,y} u_\alpha \quad (10a)$$

$$v_{,x} = N_{\alpha,x} v_\alpha \quad v_{,y} = N_{\alpha,y} v_\alpha \quad (10b)$$

Substituting Equations 6a-c and 10a-b into Equations 8c, 9a-b leads to the finite element equations that can be written in tensor form as,

$$\begin{aligned} K_{\alpha\beta\gamma} u_\beta u_\gamma + K_{\alpha\beta\gamma} v_\beta v_\gamma - \frac{1}{\rho} H_{\alpha\lambda} p_\lambda + S_{\alpha\beta} u_\beta \\ + S_{\alpha\beta} v_\beta = Q_\alpha \end{aligned} \quad (11a)$$

$$\begin{aligned} K_{\alpha\beta\gamma} u_\beta v_\gamma + K_{\alpha\beta\gamma} v_\beta v_\gamma - \frac{1}{\rho} H_{\alpha\lambda} p_\lambda + S_{\alpha\beta} v_\beta \\ + S_{\alpha\beta} v_\beta = Q_\alpha \end{aligned} \quad (11b)$$

$$H_{\beta\mu} u_\beta + H_{\beta\mu} v_\beta = 0 \quad (11c)$$

where the coefficients in these equations are defined by,

$$K_{\alpha\beta\gamma^x} = \int_A N_\alpha N_\beta N_{\gamma,x} dA \quad (12a)$$

$$K_{\alpha\beta\gamma^y} = \int_A N_\alpha N_\beta N_{\gamma,y} dA \quad (12b)$$

$$H_{\alpha\lambda^x} = \int_A N_{\alpha,x} H_\lambda dA \quad (12c)$$

$$H_{\alpha\lambda^y} = \int_A N_{\alpha,y} H_\lambda dA \quad (12d)$$

$$S_{\alpha\beta^{xx}} = \int_A 2vN_{\alpha,x} N_{\beta,x} dA + \int_A vN_{\alpha,y} N_{\beta,y} dA \quad (12e)$$

$$S_{\alpha\beta^{xy}} = \int_A vN_{\alpha,y} N_{\beta,x} dA \quad (12f)$$

$$S_{\alpha\beta^{yx}} = \int_A vN_{\alpha,x} N_{\beta,y} dA \quad (12g)$$

$$S_{\alpha\beta^{yy}} = \int_A vN_{\alpha,x} N_{\beta,x} dA + \int_A 2vN_{\alpha,y} N_{\beta,y} dA \quad (12h)$$

$$Q_{\alpha^x} = \int_{S_2} N_i T_x dS \quad (12i)$$

$$Q_{\alpha^y} = \int_{S_2} N_i T_y dS \quad (12j)$$

2.3 Computational procedure

The finite element equations, as shown in Equations 11a-c, are nonlinear to be solved by an iterative procedure. The Newton-Raphson iteration method is selected in this study. The method requires writing the unbalance values in the form,

$$\begin{aligned} F_{\alpha^x} = & K_{\alpha\beta\gamma^x} u_\beta u_\gamma + K_{\alpha\beta\gamma^y} v_\beta u_\gamma - \frac{1}{\rho} H_{\alpha\lambda^x} p_\lambda \\ & + S_{\alpha\beta^{xx}} u_\beta + S_{\alpha\beta^{xy}} v_\beta - Q_{\alpha^x} \end{aligned} \quad (13a)$$

$$\begin{aligned} F_{\alpha^y} = & K_{\alpha\beta\gamma^x} u_\beta v_\gamma + K_{\alpha\beta\gamma^y} v_\beta v_\gamma - \frac{1}{\rho} H_{\alpha\lambda^y} p_\lambda \\ & + S_{\alpha\beta^{yx}} u_\beta + S_{\alpha\beta^{yy}} v_\beta - Q_{\alpha^y} \end{aligned} \quad (13b)$$

$$F_\mu = H_{\beta\mu^x} u_\beta + H_{\beta\mu^y} v_\beta \quad (13c)$$

Then application of the method leads to a set of algebraic equations with incremental unknowns in the form,

$$\begin{bmatrix} G_{\alpha\beta^x} & L_{\alpha\beta^y} & -\frac{1}{\rho} H_{\alpha\lambda^x} \\ L_{\alpha\beta^x} & G_{\alpha\beta^y} & -\frac{1}{\rho} H_{\alpha\lambda^y} \\ H_{\beta\mu^x} & H_{\beta\mu^y} & 0 \end{bmatrix} \begin{Bmatrix} \Delta u_\beta \\ \Delta v_\beta \\ \Delta p_\beta \end{Bmatrix} = \begin{Bmatrix} F_{\alpha^x} \\ F_{\alpha^y} \\ F_\mu \end{Bmatrix} \quad (14)$$

where the coefficients in the above equations are,

$$G_{\alpha\beta x} = K_{\alpha\beta\gamma x} u_\gamma + K_{\alpha\gamma\beta x} u_\gamma + K_{\alpha\gamma\beta y} v_\gamma + S_{\alpha\beta xx} \quad (15a)$$

$$G_{\alpha\beta y} = K_{\alpha\beta\gamma y} v_\gamma + K_{\alpha\gamma\beta y} v_\gamma + K_{\alpha\gamma\beta x} u_\gamma + S_{\alpha\beta yy} \quad (15b)$$

$$L_{\alpha\beta x} = K_{\alpha\beta\gamma x} v_\gamma + S_{\alpha\beta yx} \quad (15c)$$

$$L_{\alpha\beta y} = K_{\alpha\beta\gamma y} u_\gamma + S_{\alpha\beta xy} \quad (15d)$$

These coefficients which are in form of element matrices can be evaluated in closed-form for triangular elements ready for computer programming. Details of the derivation for these element matrices are omitted herein for brevity. In these Equations 15a-d, u_γ and v_γ are the values of the velocity components at the i^{th} iteration. The iteration process is terminated if the change in percentage of the overall errors of the nodal unknowns from the previous iteration is less than the specified value. The closed form of the finite element matrices derived and the iteration procedure described are used in the development of a computer program that can be executed on standard personal computers. The main objectives in development of this computer program is such that it follows the formulation derived, easy to understand, and used together with the adaptive meshing technique which is explained in the next section. Both the finite element matrices derived and the computer program developed have been verified by a number of examples that have exact solutions prior to applying to solve more complex problems.

2.4 Adaptive meshing technique

The idea behind the adaptive meshing technique presented herein is to construct a new mesh based on the solution obtained from the previous mesh. The new mesh will consist small elements in the regions with large change in solution gradients and large elements in the other regions where the change in solution gradient is small. As an example of a flow past a cylinder, small elements are needed near the cylinder surface to capture detailed flow field, whereas larger elements can be used in the region far away from the cylinder because the flow behaviour is almost uniform. To determine proper element sizes at different locations in the flow field, the solid-mechanics concept of determining the principal stresses from a given state of stresses at a point is employed. Since small elements are needed in the regions of complex flow behaviour, the velocity distribution can be used as an indicator in the determination of proper element sizes.

To determine proper element sizes, the second derivatives of the flow velocity with respect to the global coordinates x and y are first computed,

$$\begin{bmatrix} \frac{\partial^2 V}{\partial x^2} & \frac{\partial^2 V}{\partial x \partial y} \\ \frac{\partial^2 V}{\partial x \partial y} & \frac{\partial^2 V}{\partial y^2} \end{bmatrix} \quad (16)$$

where V is the magnitude of the two velocity components u and v ,

$$V = \sqrt{u^2 + v^2} \quad (17)$$

The principal quantities in the principal directions X and Y where the cross derivatives vanish, are then determined,

$$\begin{bmatrix} \frac{\partial^2 V}{\partial X^2} & 0 \\ 0 & \frac{\partial^2 V}{\partial Y^2} \end{bmatrix} \quad (18)$$

This value is used to compute proper element size h at that locations from the conditions¹⁴,

$$\lambda = \max \left(\left| \frac{\partial^2 V}{\partial X^2} \right|, \left| \frac{\partial^2 V}{\partial Y^2} \right| \right) \quad (19)$$

This value is used to compute proper element size h at that locations from the conditions¹⁴,

$$h^2 \lambda = \text{constant} = h_{\min}^2 \lambda_{\max} \quad (20)$$

where $h_{\min}$ is the specified minimum element size, and $\lambda_{\max}$ is the maximum principal quantity for the entire model. Based on the condition shown in Equation 20 proper element sizes are generated according to the given minimum element size $h_{\min}$.

Specifying too small $h_{\min}$ may result in a model with an excessive number of elements.

On the other hand, specifying too large $h_{\min}$ may result inadequate solution accuracy or excessive analysis and remeshing cycles. These factors must be considered prior to generating a new mesh.

3. EXAMPLES

Three examples are presented to evaluate the capability of the finite element formulation combining with the adaptive meshing technique for analysis of incompressible flows. The three examples are flow past a cylinder, flow in a sudden enlargement duct, and flow through an orifice. All computations were made on standard personal computer with Pentium 166 MHz processor.

3.1 Flow past a cylinder

To demonstrate the capability of the adaptive meshing technique combining with the finite element method developed for viscous incompressible flow analysis, the problem

of flow past a cylinder¹⁵ is selected. The flow enters through the left boundary edge with the Reynolds number of 50 as shown in Figure 1. Due to the symmetry of the flow solution, only half of the domain is analyzed. The adaptive meshing technique starts from creating a relatively uniform mesh as shown in Figure 2. This initial mesh consists of 1 433 elements. Such a large number of elements is needed because, for a given flow problem in general, the flow solution is not known a priori. A large number of small elements are normally used to capture meaningful flow behavior. Element sizes for this problem are small in order to adequately represent the round surface of the cylinder. The predicted velocity distribution is shown in Figure 3. Figure 4 shows the change in the flow magnitude and direction near the cylinder surface with flow recirculation occurs behind the cylinder. These figures suggest that smaller elements are needed near the cylinder surface to capture accurate flow behavior. At the same time, larger elements can be used in the region far away from the cylinder to reduce the total number of unknowns, the computational time and the computer memory.

The flow velocity distribution obtained from the initial mesh as shown in Figure 3 is then used to determine new element sizes for generating the second adaptive as described in Section 2.4. The second adaptive mesh consisting of 583 elements is shown in Figure 5, where smaller elements are generated in the region near the cylinder surface where large change in velocity gradients occurs. At the same time, larger elements are generated in other regions away from the cylinder. With this second adaptive mesh, the finite element analysis is then performed and the corresponding flow velocity solution is shown in Figure 6.

The entire procedure is repeated again to generate the third adaptive mesh with 671 elements as shown in Figure 7. The corresponding flow solution of velocity distribution is shown in Figure 8. Detailed flow velocities near the cylinder surface is shown in Figure 9. The figure shows that the adaptive finite element mesh can provide improved flow behavior near top cylinder surface including detailed flow recirculation behind the cylinder as compared to the results from the initial uniform mesh. Figure 10 also highlights the velocity profile along the section A-A (see Figure 9) on top of the cylinder surface. The figure indicates a smooth velocity profile is obtained due to small elements are generated in this critical boundary layer for this adaptive mesh.

3.2 Flow in a sudden enlargement duct

To demonstrate the adaptive meshing capability for viscous incompressible flow, the flow behavior in a sudden enlargement duct¹⁶ is analysed. The flow enters with a parabolic profile at Reynolds number of 100 through a channel of a duct with sudden enlargement as shown in Figure 11. The first adaptive mesh consisting of 1,490 elements with relatively uniform element sizes is first generated as shown in Figure 12. The predicted velocity distribution from this mesh is shown in Figure 13. With this solution, the second adaptive mesh is then constructed as shown in Figure 14. The mesh consists only 780 elements which is approximately a half of those used in the first mesh. The corresponding predicted velocity distribution is shown in Figure 15. The figure shows improved resolution of the velocity distribution because smaller

elements are constructed in the region of large change in the velocity gradients whereas larger elements are generated in the other regions in order to reduce the computational time and computer memory. The entire process is repeated to generate the third mesh that consists of 1,100 elements as shown in Figure 16. The predicted velocity and pressure distributions from this third adaptive mesh are shown in Figures 17 and 18, respectively. These figures show improved flow solutions that include detailed flow velocity behavior with the region of flow recirculation beneath the flow inlet.

3.3 Flow through an orifice

To further demonstrate the adaptive meshing capability for capturing a more complex flow behaviour, the problem of flow through an orifice is analysed. The flow enters from the left boundary of an orifice with the Reynolds number of 50 as shown in Figure 19. Due to symmetry of the flow solution, only the upper or the lower half of the orifice can be analysed. Figure 20 shows the first adaptive mesh on the upper half that consists of 1,638 elements with relatively uniform element sizes. The figure also shows the predicted velocity distribution on the lower half. Detailed finite element mesh and velocity distributions near the orifice are shown in Figure 21 with an enlargement in Figure 22. With the predicted velocity distribution from the first mesh, the second adaptive mesh is constructed and the flow solution is carried out as shown in Figure 23. This second adaptive mesh consists of only 701 elements with small elements automatically generated near the orifice where the large change in velocity gradients occurs. The entire process is then repeated to generate the third adaptive mesh with 1,620 elements as shown in Figure 25. Even though the total number of elements generated in this third mesh is about the same as the first uniform mesh, the corresponding flow solution as shown in Figure 25 can provide detailed flow resolution near the orifice. Detailed flow behaviors as shown in Figures 26-27 indicate the increased flow velocity vectors that merge with flow from recirculation behind the orifice. Figure 28 shows the predicted pressure distribution from the third adaptive mesh. The predicted flow solutions from this third adaptive mesh demonstrates the capability of the combined adaptive meshing technique and the finite element method that can provide insight and help analysts to increase understanding of the complex flow behavior.

4. CONCLUSION

An adaptive meshing technique is combined with a finite element method for analysis of viscous incompressible flow problems. The finite element formulation and its computational procedure are first described. The Navier-Stokes equations consisting of the conservation of mass and momentums are solved. The corresponding finite element equations are derived and a corresponding computer program that can be executed on standard personal computers has been developed. The finite element method is combined with an adaptive meshing technique to improve the flow solution accuracy. The adaptive meshing technique generates an entirely new mesh based on the solution obtained from a previous mesh. The new mesh consists of clustered

elements in the regions with large change in the velocity gradients to provide higher solution accuracy. At the same time, larger elements are generated in the other regions to reduce the computational time and computer memory.

The efficiency of the combined adaptive finite element method has been evaluated by the three viscous incompressible flow problems of flow past cylinder, flow in a sudden enlargement duct, and flow through an orifice. Results from these examples demonstrate that the combined method can provide detailed flow solution behaviours at a reduced computational time and computer memory.

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Figure 1. Flow past a cylinder.

Figure 2. Initial finite element mesh for flow past a cylinder.

Figure 3. Predicted flow velocity distribution from the initial finite element mesh.

Figure 4. Detailed flow behavior near cylinder surface (insert A in Figure 3 the initial mesh).

Figure 5. The second adaptive finite element mesh for flow past a cylinder.

Figure 6. Predicted flow velocity distribution from the second adaptive finite element mesh,

Figure 7. The third adaptive finite element mesh for flow past a cylinder.

Figure 8. Predicted flow velocity distribution from the third adaptive finite element mesh.

Figure 9. Detailed flow behavior near cylinder surface (insert B in Figure 8) for the third adaptive mesh.

Figure 10. Flow velocity profile along section A-A in Figure 7.

Figure 11. Flow in a sudden enlargement duct.

Figure 12. First adaptive mesh for flow in a sudden enlargement duct.

Figure 13. Predicted velocity distribution from the first adaptive mesh.

Figure 14. Second Adaptive mesh for flow in a sudden enlargement duct.

Figure 15. Predicted velocity distribution from the second adaptive mesh.

Figure 16. Third adaptive mesh for flow in a sudden enlargement duct.

Figure 17. Predicted velocity distribution from the third adaptive mesh.

Figure 18. Predicted pressure distribution from the third adaptive mesh.

Figure 19. Flow through an orifice.

Figure 20. First adaptive mesh and predicted velocity distribution for flow through an orifice.

Figure 21. Detailed mesh and predicted velocity distribution of insert A in Figure 20.

Figure 22. Detailed velocity distribution near the orifice of insert B in Figure 21.

Figure 23. Second adaptive mesh and predicted velocity distribution for flow through an orifice.

Figure 24. Third adaptive mesh and predicted velocity distribution for flow through an orifice.

Figure 25. Detailed mesh and predicted velocity distribution of insert C in Figure 24.

Figure 26. Detailed velocity distribution of insert D in Figure 25.

Figure 27. Detailed velocity distribution of insert E in Figure 26.

Figure 28. Predicted pressure distribution from the third adaptive mesh.