COMPUTATION OF BEARING CAPACITY FACTOR N_q – TERZAGHI AND PRANDTL MECHANISM

D.M. Dewaikar, B.G. Mohapatro

Civil Engineering Department,
Indian Institute of Technology Bombay, Powai, Mumbai- 400 076, India

V.A. Sawant*

Department of Civil Engineering, Indian Institute of Technology Roorkee, India

H.S. Chore

Civil Engineering Department,
Datta Meghe College of Engineering, New Mumbai - 400 708, India

Received 23 February 2007

ABSTRACT

The unique values of the bearing capacity factors, N_q are obtained in respect of failure mechanisms suggested by Terzaghi and Prandtl using limit equilibrium approach. In this paper an attempt has been made to present the analysis integrating the Kotter's equation with Terzaghi's and Prandtl's mechanisms using limit equilibrium conditions. The analysis presented here with some modifications in the exiting analyses computes the bearing capacity factor – N_q , the expressions of which are similar to the one given by Terzaghi [10] and Reissner [7] who had extended the pioneering work initiated by Prandtl [6]. The analysis reported here successfully demonstrates the applicability of Kotter's equation to the solution of limit equilibrium problems involving plane or curved failure surfaces.

Keywords: Limit Equilibrium Condition, Kotter's Equation, Terzaghi Mechanism, Prandtl Mechanism, Passive Thrust, Bearing Capacity Factor

1. INTRODUCTION

Problems of ultimate bearing capacity deal with the conditions for the equilibrium of soil immediately preceding ultimate failure by plastic flow. These problems can be solved using theory of plasticity. In order to solve these problems, the stress conditions for the failure of the soil are sufficient and consideration of state of strain is not required.

Analysis of ultimate bearing capacity represents a primary step for evaluation of the stability of shallow foundations. Even in the analysis of settlements, which is normally carried out to ascertain whether the foundation can fulfill their intended function from the structural point of

* Corresponding author e-mail: vasawant@yahoo.com

view, computation of bearing capacity continues to be the most important component for the design of foundations. Many researchers have contributed for the analysis of ultimate bearing capacity of shallow foundations using various methods and techniques. Slip line, limit equilibrium, limit analysis and finite element analysis are the important methods used for the analysis of bearing capacity of shallow foundations. The initial contributions in this area were made by Prandtl [6] and Reissner [7].

Kotter [3] was the first to derive slip line equations for the case of plane deformations. Using these equations, Prandtl [6] obtained an analytical closed form solution for the ultimate bearing pressure for the case of strip footing on weightless semi-infinite space. His analysis is applicable to frictional cohesive soil and to purely cohesive soil. Reissner [7] extended his work for case of purely frictional weightless material with the surface of the half space (except for the strip) loaded by a uniform surcharge and obtained closed form solution of this analysis. Terzaghi [10] proposed the first comprehensive bearing capacity analysis for the case of strip footing with rough base for a frictional cohesive ponderable soil using limit equilibrium method. Treating the soil lying above the footing base as surcharge with no shear strength and using the principle of superposition, Terzaghi evaluated ultimate bearing pressure on the footing in terms non-dimensional bearing capacity factors which are purely dependent upon the angle of internal friction of soil. Meyerhof [4] used limit equilibrium method for the evaluation of ultimate bearing capacity of shallow foundation with rough base for cohesive frictional ponderable soil and proposed a bearing capacity equation similar to that of Terzaghi, but the bearing capacity factors were different. Hansen [2] used quasi-empirical modifications of slip line solutions for evaluating bearing capacity of shallow foundations. The contributions by Chen [1], Saran and Agrawal [8] and Soubra [9] in the evaluation of bearing capacity of shallow foundations are also noteworthy.

The bearing capacity factors, N_c and N_q are uniquely determined in respect of failure mechanisms of Terzaghi and Prandtl using limit equilibrium approach. In this paper an attempt has been made to present the analysis based on Kotter's equation and bearing capacity factor N_q has been determined considering Terzaghi's and Prandtl's mechanisms using limit equilibrium conditions with some modifications in the exiting analyses.

2. OUTLINE OF THE PROPOSED ANALYSIS

In Fig. 1 and Fig. 2, the failure mechanisms for a strip footing are shown, in which, the wedge angle, $\psi = \phi$ for Terzaghi's mechanism and $\psi = \pi/4 + \phi/2$ for Prandtl's mechanism. Considering the free body diagram of the wedge, CDFB, (Fig.3), the following forces are considered:

Fig. 1: Failure mechanism Prandtl's analysis.

Fig. 2: Failure mechanism Terzaghi's analysis

Fig. 3: Free-body diagram of wedge CDFB.

Q_t = Surcharge acting on BF and intensity of surcharge is q , which is uniformly distributed over the length BE.

P_q = Passive Thrust due to surcharge on EF exerted on the Rankine's wall, DF and acting at a distance of $(DF/2)$ from point, D.

R = Soil Reaction on the curved part, CD of the failure surface.

P_{pq} = Passive thrust due to surcharge acting on the face CB at point G, and

This analysis is primarily based on the computation of vertical (R_V) and horizontal (R_H) components of the resultant soil reaction, R that acts on the curved part, CD of the failure surface (Fig.3). For this purpose, Kotter's equation is used. After computing these components, free body diagram of the wedge, CDFB, is considered. In this free body diagram, known forces are R_H , R_V , P_q , Q_t , and unknown force is passive thrusts, P_{pq} .

Kotter's Equation

For a cohesionless soil medium with angle of internal friction Φ in passive state of equilibrium,

Kotter's equation gives a solution for determining the distribution of soil pressure p along the arc of the failure surface in the following form (Fig. 4):

Fig. 4: Kotter's equation for cohesionless soil in passive state of equilibrium.

$$\frac{dp}{ds} + 2p \tan \phi \frac{d\alpha}{ds} - \gamma \sin(\alpha + \phi) = 0 \quad \dots(1)$$

In which dp is differential reactive pressure on the elemental length ds of the failure surface, and α is angle made by the tangent to failure surface at the point of interest with the horizontal. For the evaluation of P_{pq} the problem is analyzed for the case $c = 0$, $\gamma = 0$, and $q > 0$. For the computation of passive thrust, P_{pq} , pole of the log spiral is considered to lie at the edge of the footing and unique value of P_p is evaluated considering two equations of force equilibrium.

To compute the point of application of P_{pq} , moment equilibrium condition is used, in which, moments of all the forces and reactions about the pole of the log spiral (Fig. 3) are considered. This is convenient since the resultant reaction R always acts along the radial line of the log spiral and makes no contribution to the moment.

Thus, in the proposed analysis, force equilibrium conditions are used to compute P_{pq} and moment equilibrium condition is used to compute its point of application.

3. GEOMETRY OF THE FAILURE SURFACE

The failure surface (Fig.1 and 2) has two parts, namely CD, being part of the log spiral and DE, its tangent. The geometry of this failure surface is described in terms of half footing width, B , angle of soil internal friction, ϕ and wedge angle, ψ .

Now, $CB = r_0$ = initial radius of the log spiral, which is defined by the following equation

$$r = r_0 e^{\theta \tan \phi} \quad (2)$$

In the above equation, r is the radius of the log spiral, for which, the angle as measured from initial radius is θ and as seen from Fig. 3, the maximum value of θ is θ_m . From the geometry of the triangular wedge, ABC, distance CB is given as:

$$CB = B \sec \psi \quad (B = \text{Half width of footing}) \quad (3)$$

From the equation of the log spiral, the distance BD or DE is given as:

$$BD = r_1 = r_0 e^{\theta_m \tan \phi} \quad (4)$$

Similarly, from the geometry of Fig. 3, the distances BF and DF are given as:

$$BF = r_1 \cos(\pi/4 - \phi/2) \quad \text{and} \quad DF = r_1 \sin(\pi/4 - \phi/2) \quad (5)$$

4. COMPUTATION OF N_q

For this computation, Fig.5 is considered, in which, the forces acting on the wedge, CDFB are shown. As seen earlier (Fig. 1 and 2), the failure surface has two parts, namely CD, being part of log spiral and, DE, its tangent.

Fig. 5: Geometrical relationships for pole below footing base.

4.1 Integration over the part DE of the Failure Surface

This part of the failure surface being straight, $d\theta/ds = 0$ and as soil is considered weightless, ($\gamma=0$), Eq. 1 reduces to the following form:

$$\frac{dp}{ds} = 0 \quad (6)$$

Integration yields, $p = \text{constant}$

Soil reaction pressure, p is constant throughout the length DE, and it is calculated considering the equilibrium of wedge, DEF. Referring to Fig.5, the resultant reaction, R' acting on the failure surface, DE is given as,

$$R' = \frac{P_q}{\cos(45 - \phi/2)} \quad (7)$$

where, P_q is the passive Rankine's thrust and its magnitude is given as ,

$$P_q = q DF K_p \quad \text{with} \quad K_p = \tan^2(45 + \phi/2) \quad (8)$$

As soil pressure p is constant through out the length DE, the value at point D, is given as,

$$p_{(D)} = \frac{R'}{DE} = q K_p \tan(45 - \phi/2) \quad (9)$$

Among the forces shown in Fig. 3, the known forces are P_q and Q_t and the forces R_H and R_V are calculated using Kotter's equation for a curved failure surface.

For a weightless soil, Kotter's equation (Eq. 1) reduces to the following forms.

$$\frac{d p}{d s} + 2 p \tan \phi \frac{d \alpha}{d s} = 0 \quad (10)$$

Multiplying throughout by ds and rearranging Equation 10 comes,

$$\frac{d p}{p} = -2 \tan \phi d \alpha \quad (11)$$

From the geometry of the log spiral (Eq. 2),

$$\frac{d s}{d \theta} = r \sec \phi \quad (12)$$

Referring to the Fig. 5, the angle α is evaluated in terms of angle θ , as given below.

$$\angle BIC = \theta_L = 90^\circ - (\psi - \phi)$$

In quadrilateral BCJM,

$$\angle BCJ = 90^\circ + \phi, \quad \angle BMJ = 90^\circ - \phi, \quad \angle CBM = \theta \quad \text{and} \quad \angle CJM = 180^\circ - \theta$$

At point L, $180^\circ = \angle CJM + \angle MJL + \angle LJK = 180^\circ - \theta + \alpha + \theta_L$

From which, $\alpha = \theta - \theta_L$, with $d\alpha = d\theta$

Substituting $d\theta$ for $d\alpha$, Eq. 11 becomes,

$$\int_p^{p_{(D)}} \frac{d p}{p} = - \int_{\theta}^{\theta_m} 2 \tan \phi d \theta \quad (13)$$

Integration yieldss the pressure distribution along the curved failure surface, CD as,

$$p = p_{(D)} e^{2(\theta_m - \theta) \tan \phi} \quad (14)$$

4.2 Components of the Resultant Reaction on the Curved Failure Surface

The resultant reaction, R on the curved failure surface is given as (Fig. 5),

$$R = \int p ds \quad (15)$$

The vertical component, R_V of the reaction is obtained as,

$$R_V = \int_0^{\theta_m} p \sin(\pi - (\psi + \theta)) ds = \int_0^{\theta_m} p \sin(\psi + \theta) r \sec \phi d\theta \quad (16)$$

After substituting the value of p from Eq. 14, R_V is obtained in the following form,

$$R_V = q K_p r_0 \cos \phi \tan(45 - \phi/2) \left[-e^{\theta_m \tan \phi} \{ \sin(\psi + \theta_m) \tan \phi + \cos(\psi + \theta_m) \} + e^{2\theta_m \tan \phi} \{ \sin \psi \tan \phi + \cos \psi \} \right] \quad (17)$$

Similarly, the horizontal component, R_H of the resultant reaction is given as (Fig.),

$$R_H = - \int_0^{\theta_m} p r_0 e^{\theta \tan \phi} \cos(\psi + \theta) \sec \phi d\theta \quad (18)$$

After substituting value of p from Eq. 14, R_H is obtained in the following form,

$$R_H = -q K_p r_0 \cos \phi \tan(45 - \phi/2) \left[e^{\theta_m \tan \phi} \{ \sin(\psi + \theta_m) - \tan \phi \cos(\psi + \theta_m) \} + e^{2\theta_m \tan \phi} \{ \cos \psi \tan \phi - \sin \psi \} \right] \quad (19)$$

4.3 Surcharge Weight, Q_t on BF

The overburden over surface BF (Fig.5) is replaced by a concentrated load of intensity, Q_t , which is given as,

$$Q_t = q(BF) = q r_0 e^{\theta_m \tan \phi} \cos(45 - \phi/2) \quad (20)$$

4.4 Computation of Passive Thrust (P_{pq})

This is obtained by using the two equations of force equilibrium (Fig. 5).

$$\text{Vertical force equilibrium} \quad R_V - P_{pq} \cos(\psi - \phi) - Q_t = 0 \quad (21)$$

$$\text{Horizontal force equilibrium} \quad P_{pq} \sin(\psi - \phi) - R_H - P_q = 0 \quad (22)$$

It may be noted that in the above equations, the only unknown is P_{pq} and therefore, if the pole of the log spiral is correctly located, both the equations should yield the same value of P_{pq} and otherwise, two different values of P_{pq} would be obtained. Eqs.21 and Eq.22 were solved for P_{pq} after introducing the values of P_q and Q_t from Eqs. 8 and Eq.20. It was found that both the equations yielded the same value of P_{pq} as given below:

$$P_{pq} = \frac{q B e^{2\theta_m \tan \phi}}{\tan(45 - \phi/2) \cos \psi} \quad (23)$$

Since the above unique value of P_{pq} is obtained for the location of log spiral at the footing edge, it is clear that this location is the correct and only possible location of the pole of log spiral, for any value of the wedge angle, ψ .

4.5 Computation of N_q

For this computation, Fig. 6 is referred, which shows the free body diagram of the triangular

wedge, ABC, subjected to forces Q_u (kN/m) and vertical component, P_{pqV} of the passive thrust, P_{pq} . The vertical force equilibrium gives:

$$Q_u = 2P_{pq} \cos(\psi - \phi) \quad (24)$$

Fig. 6: Free-body Diagram of Triangular Wedge ABC for Determination of N_q .

Dividing throughout by $2B$ (Width of footing),

$$\frac{Q_u}{2B} = q_u = \frac{2P_{pq} \cos(\psi - \phi)}{2B} = q N_q \quad (25)$$

$$N_q = \frac{P_{pq} \cos(\psi - \phi)}{q B} = \frac{q B e^{2\theta_m \tan \phi}}{\tan(45 - \phi/2) \cos \psi} \times \frac{\cos(\psi - \phi)}{q B} = \frac{\cos(\psi - \phi) e^{2\theta_m \tan \phi}}{\tan(45 - \phi/2) \cos \psi} \quad (26)$$

4.6 Point of application of P_{pq}

The passive thrust, P_{pq} acts on the line CB at an angle, ϕ to the normal at G with distance BG from pole B (Fig.5). Taking moment of all forces about pole B, the following equation is obtained:

$$P_{pq} \sin(\psi - \phi) BG \sin \psi + P_{pq} \cos(\psi - \phi) BG \cos \psi = Q_u (BF/2) + P_q (DF/2) \quad (27)$$

After simplification

$$\begin{aligned}
 P_{pq} \quad BG \cos \phi &= q \quad BF \left(BF / 2 \right) + q \quad DF \quad K_p \left(DF / 2 \right) = q \left(BF \right)^2 \\
 \frac{q \quad B \quad e^{2\theta_m \tan \phi}}{\tan(45 - \phi / 2) \cos \psi} \times BG \cos \phi &= q \quad B^2 \sec^2 \psi \quad e^{2\theta_m \tan \phi} \cos^2(45 - \phi / 2) \\
 BG \cos \phi &= B \sec \psi \times \sin(45 - \phi / 2) \cos(45 - \phi / 2) = B \sec \psi \times \frac{\sin(90 - \phi)}{2} \\
 BG \cos \phi &= B \sec \psi \times \frac{\cos \phi}{2} = 0.5 \quad BC
 \end{aligned} \tag{28}$$

The computation shows that the point of application of P_{pq} is located at midway of the face BC.

5. RESULTS AND DISCUSSIONS

Closed form solutions are obtained from the present analysis for bearing capacity factor, N_q considering mechanisms postulated by Terzaghi and Prandtl. For Terzaghi mechanism, replacing the wedge angle, ψ with ϕ in the general solution for N_q , the following expression is obtained.

$$N_q = \frac{e^{2(3\pi/4 - \phi/2) \tan \phi}}{2 \cos^2(45 + \phi/2)} \tag{29}$$

The above expression is similar to Terzaghi's (1943) expression.

For Prandtl mechanism, replacing the wedge angle, $\psi = \pi/4 + \phi/2$ in the general solution for N_q , the following expression is obtained.

$$N_q = e^{\pi \tan \phi} \tan^2(45 + \phi/2) \tag{30}$$

The above expression is similar to that given by Reissner (1923).

The use of Kotter's equation enables the application of moment equilibrium condition from which, point of application of passive thrust, P_{pq} is at the centre of BC for all values of ϕ .

6. CONCLUSION

Thus, from the above analysis, it is clearly seen that Kotter's equation is applicable to the solution of limit equilibrium problems involving plane or curved failure surfaces. The concept of force equilibrium condition identifies a unique failure surface consistent with the specific failure mechanism, which finally gives the passive thrust. The analysis also shows that pole of the log spiral is located at the footing edge in respect of evaluation of bearing capacity factors, N_q , for both Terzaghi's and Prandtl's mechanisms.

Another advantage of Kotter's equation is that it enables the use of moment equilibrium condition from which the point of application of passive thrust can be computed. This is not possible in any of the exiting limit equilibrium analyses, in which, an assumption regarding point of application is required to be made so as to make the analysis determinate.

REFERENCES

1. Chen, W.F. (1975,)Limit Analysis and Soil Plasticity, Elsevier Scientific Publishing

Company, London.

2. Hansen, J.B. (1961), A General Formula for Bearing capacity, Danish Geotechnical Institute, Copenhagen, Denmark, Bulletin, 11, pp. 38-46.
3. Kotter, F. (1903), Die Bestimmung des Drucks angekrummten, eine Aufgabe aus der Lehre vom Erddruck, Sitzungsberichte der Akademie der Wissenschaften, Berlin (in German), pp. 229-233.
4. Meyerhof, G.G. (1951), The Ultimate Bearing Capacity of Foundations, *Geotechnique*, 2, pp. 301-332.
5. Mohapatro, B.G. (2001), Some Studies on Bearing Capacity of Shallow Foundations, Ph.D. Thesis, IIT Bombay, India.
6. Prandtl, L. (1920), Über die Harte plastischer Körper. Nachrichten von der Königl. Gesellschaft der Wissenschaften zu Göttingen, Mathematisch-Physikalische Klasse aus dem Jahre, Berlin (in German), pp. 74-85.
7. Reissner H. Zum Erddruckproblem (1924), Proceedings of the International Congress for Applied Mechanics, Delft, The Netherlands (in German), pp. 295-311.
8. Saran, S. and Agarwal, R.K. (1991), Bearing Capacity of Eccentrically Obliquely Loaded Footing, *Journal of Geotechnical Engineering*, ASCE, 117(11), pp. 1669-1690.
9. Soubra, A.H. (1999), Upper Bound Solutions for Bearing Capacity of Foundations, *Journal of Geotechnical and Geoenvironmental Engineering*, ASCE, 125(1), pp. 59-68 .
10. Terzaghi, K. (1943), *Theoretical Soil Mechanics*, Wiley, New York, USA.