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NUMERICAL RESULT FOR VARIATIONAL INEQUALITY PROBLEMS WITH EQUALITY CONSTRAINT

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ABSTRACT

In this paper we present some numerical results for illustrating a theoretical results obtained in our investigation in the field of variational inequality problems with constraint in the form of operator equation involving monotone operator.

Key words: Monotone operators, Fréchet differentiable, weakly lower semi-continuous proper convex functional, and regularization.

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1. INTRODUCTION

Many problems arising in mathematical physics and mechanics have been formulated in the following abstract form: find an element $x \in S_0$ such that

$$\varphi(x_0) = \min_{x \in S_0} \varphi(x) \tag{1.1}$$

where φ is a weakly lower semicontinuous and properly convex functional on a real re°exive Banach space X with the norm denoted by $\|.\|$, and S_0 is a convex and closed subset of the space X. If we denote by $A(x)$ the subdifferential of the functional φ at the point x, then problem (1.1) is equivalent to the variational inequality

$$\langle A(x), x - x_0 \rangle \geq 0 \qquad \forall x \in S_0, \quad x_0 \in S_0, \tag{1.2}$$

where the symbol $\langle x^*, x \rangle$ denote the value of the linear and continuous functional $x^* \in X^*$ at the point $x \in X$, and X^* is the adjoint space of the space X. For the sake of simplicity, the norm of X will be denoted by the symbol $\|.\|$, too. When φ is a convex functional and S_0 is the set of solutions of the operator equation

$$F(x) = f_0, \qquad f_0 \in X^*, \tag{1.3}$$

where $F: X \to X^*$ is a monotone and continuous operator, and $f_0 \in R(F)$, the range of F, it is well known (see [1]) that Problem (1.3), without additional conditions on the structure of F such as strong or uniform monotonicity, is ill-posed. By this we mean that solutions of (1.3) do not depend continuously on the data (F, f_0). So, Problem (1.2)-(1.3) in this case is ill-posed, too. In numerical computation, we often know the approximations (F_h, f_δ) of (F, f_0) such that

$$\|F(x) - F_h(x)\| \le hg(\|x\|), \quad \forall x \in X, \quad \|f_\delta - f_0\| < \delta, \quad h, \delta \to 0, \tag{1.4}$$

where $g(t)$ is some real continuous and positive function. Assume that A, the Fréchet derivative of φ satisfies the condition

$$\langle A(x) - A(y), x - y \rangle \ge m_A \|x - y\|^s \qquad \forall x, y \in X, \quad m_A > 0, \tag{1.5}$$

for $s \in R, \quad 2 \le s < +\infty$.

2. THEORETICAL RESULTS

A regularized solution of the stated problem can be constructed by the following operator equation (see[3])

$$F_h(x) \; + \; \alpha A(x) = f_\delta \qquad \alpha > 0. \tag{2.1}$$

Assumption 2.1 There is a positive constant $\tilde{r}$ such that

$$\|F(y) - F_h(x) - F'(x)(y - x)\| \le \tilde{r}\|F(y) - F(x)\|,$$

for y belonging to some neighbourhood of S_0, $x \in S_0$.

We have the following results (see [3], [4]).

Theorem 2.1 *Assume that the following conditions hold:*

(i) F *is Fréchet differentiable at some neighbourhood of* S_0 *with Assumption 2.1, for* $x = x_0$*;*

(ii) there exists an elements $z \in X$ *such that*

$F'(x_0)^* z = A(x_0)$*;*

(iii) the parameter α *is chosen such that* $\alpha \sim (\delta + h)^\rho, \quad 0 < \rho < 1$.

Then,

$$\|x_\alpha^r - x_0\| = O((\delta + h)^{\beta_1}),$$

$$\beta_1 = \min\left\{ \frac{1-\rho}{s-1}, \frac{\mu}{s} \right\}.$$

For computation, consider the finite-dimensional problems

$$F_h^n(x_{\alpha,n}^r) + \alpha A^n(x_{\alpha,n}^r) = f_\delta^n \quad ; \tag{2.2}$$

with $F_h^n = P_n^* F_h P_n$, $A^n = P_n^* A P_n$, $f_h^n = P_n^* f_\delta$, where $P_n : X \to X_n$, denotes the linear projection from X onto its subspace X_n satisfying the condition

$$X_n \subset X_{n+1}, \quad P_n x \to x, \quad n \to +\infty, \quad \forall x \in X,$$

and P_n^* is the adjoint of P_n, $\|P_n\| \le c_0$, c_0 is a positive constant. Without loss of generality, suppose that $\|P_n\| = 1$.

Set

$$\gamma_n(x) = \|(I - P_n)x\|, \quad x \in X, \quad \gamma_n^*(f) = \|(I^* - P_n^*)f\|$$

$$\gamma_n = \max\{\gamma_n(x), \quad x \in S_0, \quad \gamma_n^*(f)\}$$

where I and I^* denote the identity operators in X and X^*, respectively. By Theorem 1.1 there exists $x_{\alpha,n}^r$ satisfying (1.7) for every $\alpha > 0$.

Suppose that

$$\|A(x) - A(y)\| \le C(R)\|x - y\|^\upsilon$$

where $C(R),\ R > 0$, is a positive increasing function on $R = \max\{\|x\|, \ \|y\|\}$ such that

$$\limsup_{t \to +\infty} \frac{C(t)}{t^{\mu_0}} \le q_2 < +\infty, \quad 0 < \mu + \mu_0 < s - 1,$$

and A is continuous with $A(0) = 0$.

The convergence of the sequence $\{x_{\alpha,n}^r\}$ to x_0 is established by the following therorem.

Theorem 2.2 *Suppose that the following conditions hold:*

(i) F *is Fréchet differentiable at some neighbourhood of* S_0 *with Assumption 2.1, for* $x = x_0$ *;*

(ii) $F_h(X_n)$ *are contained in* X_n^* *for suficiently large* n *and small* h *;*

(iii) there exists an elements $z \in X$ *such that*

$$F'(x_0)^* z = A(x_0);$$

(iv) the parameter α *is chosen such that* $\alpha \sim (\delta + h + \gamma_n)^\rho, \quad 0 < \rho < 1.$

Then, we have

$$\left\| x_{\alpha,n}^{r} - x_0 \right\| = O((\delta + h + \gamma_n)^{\beta_1} + \gamma_n^{\mu/(s-1)}) .$$

3. NUMERICAL EXAMPLE

For illustration, we consider problems (1.2) - (1.3) when

$$\varphi(x) = \frac{1}{2}\left\| x - x_* \right\|_{L_2[0;1]}^2 \tag{3.1}$$

and F is the subdifferentional of the functional $\psi(x)$, $\psi(x) = \tilde{\psi}(\langle Kx, x\rangle)$, where

$$\tilde{\psi}(t) = \begin{cases} 0 & if \quad t \le a_0; \\ \dfrac{(t-a_0)^2}{2h} & if \quad a_0 < t \le a_0 + h; \\ t - a_0 - \dfrac{h}{2} & if \quad a_0 + h < t; \end{cases} \tag{3.2}$$

h is positive parameter, a_0 is fixed positive number,

$$Kx(t) = \int_0^1 k(t,s)x(s)ds, \qquad x(s) \in L_2[0;1]$$

with

$$k(x,s) = \begin{cases} t(1-s) & if \quad t \ge s; \\ s(1-t) & if \quad t < s. \end{cases}$$

Then, K is the linear, continuous and nonnegative operator on $L_2[0;1]$. Since $\langle Kx, x\rangle$ is a convex functional and $\psi(t)$ is a convex function then $\psi(\langle Kx, x\rangle)$ is a convex functional, too. Therefore $F(x) = \partial\psi(x)$ is monotone operator and

$$F(x) = \partial\psi(x) = \frac{1}{2}\tilde{\psi}'(\langle Kx, x\rangle)Kx . \tag{3.3}$$

From (3.2) and (3.3) it follows

$$F(x) = \begin{cases} 0 & if \quad \langle Kx, x\rangle \le a_0; \\ \dfrac{\langle Kx, x\rangle}{h} Kx & if \quad a_0 < \langle Kx, x\rangle \le a_0 + h; \\ Kx & if \quad a_0 + h < \langle Kx, x\rangle. \end{cases}$$

If choosing $x_* = 0$, then solution of problem (1.3) is $x_* = 0$.

In numerical computation, we choose method as follows:

+) Divide [0; 1] in to n equal parts by divide points

$$t_i = ih;\quad h = \frac{1}{n};\quad i = 0, 1, .., n.$$

+) $\int_0^1 k(t,s)x(s)ds$ replaced by

$$T_i = \sum_{j=0}^{n} b_j k(t_i, t_j) x(t_j);\qquad i = 0, 1, .., n,$$

where $b_0 = b_n = \frac{h}{2}$, $b_1 = b_2 = \;.\;.= b_{n-1} = h$.

+) $\int_0^1\int_0^1 k(t,s)x(s)x(t)\,ds\,dt$ replaced by

$$T = \sum_{i=0}^{n} b_i \left(\sum_{j=0}^{n} b_j k(t_i, t_j) x(t_j) \right) x(t_i).$$

Problem (3.1) is appproximated by

$$F\left(x(t_i)\right) = \begin{cases} 0 & if \quad T \le a_0; \\ \frac{T}{h} T_i & if \quad a_0 < T \le a_0 + h; \\ T_i & if \quad a_0 + h < T. \end{cases} \tag{3.4}$$

($i = 0, 1, .., n$)

Problem (3.4) is an algebraic equations systems ($n+1$ equations). To solve problem (3.4), we use the iterative regularization method as follows (see [2])

$$z_{k+1} = z_k - \beta_k (F_h^n(z_k) + \alpha_k z_k) \qquad k = 0, 1, \ldots \tag{3.5}$$

with $\beta_k = (1+k)^{-1/2}$; $\alpha_k = (1+k)^{-p}$, $0 < p < 1$, choosing $p = \frac{1}{4}$.

We apply the iterative regularization method according to (2.5) to find approximative solutions as follows:

1^0) Choosing $x = (x_0, x_1, .., x_n) \ne (0, 0, .., 0)$, a_0, h.

2^0) Computing T_n and T.

3^0) Check conditions according to T:

+) If $T \le a_0$, $x = 0$ and stop iterative steps;

+) If $T > a_0 + h$, $x = x - \beta_k (Kx + \alpha_k z_k) = x - e$, moving to step 4^0;

+) If $a_0 < T \le a_0 + h$, $x = x - \beta_k \left(\frac{T - a_0}{h} Kx - \alpha_k z_k \right) = x - e$, moving to step 4^0.

4^0) Check $\|e\| \le \varepsilon$?

+) If it is true, x is approximative to the need, and then stop.

+) If it is false, moving to step 2^0.

Result of computation are showed in table 1:

Table 1.

k	t = 0.0	t = 0.2	t = 0.4	t = 0.6	t = 0.8	t = 1.0
1	0.0323223305	0.0323164376	0.0323134912	0.0323134912	0.0323164376	0.0323223305
2	0.0226556990	0.0226501456	0.0226473689	0.0226473689	0.0226501456	0.0226556990
3	0.0167460181	0.0167414661	0.0167391902	0.0167391902	0.0167414661	0.0167460181
4	0.0128547780	0.0128511166	0.0128492859	0.0128492859	0.0128511166	0.0128547780
5	0.0101523938	0.0101494314	0.0101479502	0.0101479502	0.0101494314	0.0101523938
6	0.0081985647	0.0081961396	0.0081949270	0.0081949270	0.0081961396	0.0081985647
7	0.0067406309	0.0067386205	0.0067376154	0.0067376154	0.0067386205	0.0067406309
8	0.0056246518	0.0056229656	0.0056221226	0.0056221226	0.0056229656	0.0056246518
9	0.0047522638	0.0047508343	0.0047501195	0.0047501195	0.0047508343	0.0047522638
10	0.0040581302	0.0040569067	0.0040562950	0.0040562950	0.0040569067	0.0040581302
11	0.0034974310	0.0034963749	0.0034958468	0.0034958468	0.0034963749	0.0034974310
12	0.0030385705	0.0030376519	0.0030371926	0.0030371926	0.0030376519	0.0030385705
13	0.0026587492	0.0026579447	0.0026575425	0.0026575425	0.0026579447	0.0026587492
14	0.0023411783	0.0023404695	0.0023401151	0.0023401151	0.0023404695	0.0023411783
15	0.0020732737	0.0020726458	0.0020723318	0.0020723318	0.0020726458	0.0020732737
16	0.0018454539	0.0018448948	0.0018446152	0.0018446152	0.0018448948	0.0018454539
17	0.0016503209	0.0016498208	0.0016495707	0.0016495707	0.0016498208	0.0016503209
18	0.0014820908	0.0014816415	0.0014814168	0.0014814168	0.0014816415	0.0014820908
19	0.0013361899	0.0013357848	0.0013355823	0.0013355823	0.0013357848	0.0013361899
20	0.0012089650	0.0012085984	0.0012084151	0.0012084151	0.0012085984	0.0012089650
21	0.0010974700	0.0010971372	0.0010969708	0.0010969708	0.0010971372	0.0010974700
22	0.0009993093	0.0009990062	0.0009988547	0.0009988547	0.0009990062	0.0009993093

Remarks. From the numerical table we have the following remarks:

- The number of iterations depends on the the choice of values of p.

- If number of iterations is large, approximation solution is near the exact solution of the original problem.

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