USING TWO TIME SCALES OF TURBULENCE FOR BOUNDARY LAYER FLOWS

Nawaf H. Saeid

School of Mechanical Engineering University Sains Malaysia,
Engineering Campus 14300, Nibong Tebal, Seberang Perai Selatan,
Pulau Pinang, Malaysia

Received 15 May 2002, Accepted 10 July 2002

ABSTRACT

A near wall improvement of the k - ϵ model of turbulence is proposed and evaluated. The present model takes into account the asymptotic behavior of turbulent scales near the wall region and for free turbulence region. The Kolmogorov turbulent time scale is introduced as a lower limit. The model is used for prediction of turbulent boundary layer flows. Predictions are compared with experimental data of several flow cases, with encouraging results.

1. INTRODUCTION

Due to its many practical engineering applications, turbulent shear flows have received increasing attention in various experimental and theoretical studies. In the computational fluid dynamics application, the differential k - ϵ model of turbulence is now regarded as powerful tool for prediction of turbulent boundary layer flows. In this model the eddy viscosity concept is used which is calculated from the velocity scale and length (or time) scale of the turbulence. In the standard model [1], turbulent flow calculations were performed using wall functions to describe the flow between the first grid point and the wall. This was necessary because the modeled equations were valid for high Reynolds number flows and, therefore, could not be applied at and near wall. The need to modify wall functions and to render the modeled turbulence equations near a wall was first recognized by Jones and Launder [2], followed by many other variants. Near wall corrections were proposed and they were usually in the form of extra terms added to the turbulent kinetic energy and its dissipation rate equation. These terms were derived to render the modeled equations valid at the wall and little consideration was given to ensure correct near wall asymptotic behavior. Moreover, the empirical constants of some of the near wall turbulence models are different from the standard model constants, does not predict accurately for high and low Reynolds number simultaneously.

It seems likely that the dissipation transport equation, used to provide a length (or time) scale in two-equation turbulence models, is the major source of error [3]. According to Rodi [4], the basic reason for this lack of universality of the k - ϵ models is the particular

model form of the source/sink term in the ϵ equation.

The present work is restricted to the turbulent boundary layer flows. A new method of modeling the eddy viscosity and the dissipation transport equation is introduced. Using two-region approach and adopting turbulent scales to each region, the amended expression of the eddy viscosity and ϵ can be obtained. In order to predict the low Reynolds number as well as the high Reynolds number flows, the standard empirical constants and turbulent scales for the high Reynolds number is used. To demonstrate the performance of the new method, the results are compared with direct numerical simulation DNS data and experimental measurements of the turbulent boundary layer flows. The agreement with the data is encouraging to extend the present method for more complex turbulent shear flows.

2. GOVERNING EQUATIONS

The time averaged (mean) equations of motion for turbulent boundary layer flow can be derived from the instantaneous continuity and Navier-Stokes equations by applying time averaging and invoking the Prandtl boundary layer hypothesis; The resultant steady two dimensional turbulent boundary layer equations can be written as:

$$\frac{\partial(\rho U)}{\partial x} + \frac{\partial(\rho V)}{\partial y} = 0 \quad (1)$$

$$\rho U \frac{\partial U}{\partial x} + \rho V \frac{\partial U}{\partial y} = \frac{\partial}{\partial y} \left[\mu \frac{\partial U}{\partial x} - \rho \overline{uv} \right] - \frac{dp}{dx} \quad (2)$$

Using the Boussinesq model, the Reynolds stress $-\rho \overline{uv}$ can be expressed in terms of the mean velocity gradient and the eddy (turbulent) viscosity μ_t as:

$$-\rho \overline{uv} = \mu_t \frac{\partial U}{\partial y} \quad (3)$$

Therefore, once the distribution of the eddy viscosity μ_t is known, equations (1-3) can be solved to give the velocity distribution inside the boundary layer for the no slip boundary condition. In the k - ϵ models the eddy viscosity is given by:

$$\mu_t = \rho C_{\mu} f_{\mu} \frac{k^2}{\epsilon} \quad (4)$$

Here k and ϵ are the turbulent kinetic energy and its dissipation rate, respectively. The governing equations for most of the near wall (Low-Reynolds number) k - ϵ models can be cast into the following form [5,6]:

$$\rho U \frac{\partial k}{\partial x} + \rho V \frac{\partial k}{\partial y} = \frac{\partial}{\partial y} \left[\left(\mu + \frac{\mu_t}{\sigma_k} \right) \frac{\partial k}{\partial y} \right] + \mu_t \left(\frac{\partial U}{\partial y} \right)^2 - \rho \epsilon \quad (5)$$

$$\rho U \frac{\partial \varepsilon}{\partial x} + \rho V \frac{\partial \varepsilon}{\partial y} = \frac{\partial}{\partial y} \left[\left(\mu + \frac{\mu_t}{\sigma_k} \right) \frac{\partial \varepsilon}{\partial y} \right] + \left[C_{\varepsilon 1} f_1 \mu_t \left(\frac{\partial U}{\partial y} \right)^2 - C_{\varepsilon 2} f_2 \rho \varepsilon \right] T_t^{-1} \quad (6)$$

Therefore the differences in the near wall (Low-Reynolds number) k - ε models are in the choice of the model constants C_μ , σ_k , σ_ε , $C_{\varepsilon 1}$, $C_{\varepsilon 2}$ and the specification of the damping functions f_μ , f_1 , f_2 . While the turbulent time scale T_t is defined in most of the models [5,6] as $T_t = k/\varepsilon$, which describes the turbulent time scale of the high-Reynolds number flows.

3. MODEL FORMULATION

The analysis of the turbulent boundary layer can be treated in a piecemeal fashion by first dealing with the surface layer (near wall region) and the rest of the flow (which is called the outer region) separately and then reconciling these partial descriptions with appropriate asymptotic method^[7]. Accordingly, the following characteristic scales of turbulence are used:

1. In near the wall region, the Kolmogorov universal equilibrium theory of the small-scale structure is used [7]. According to this theory, the parameters governing the small-scale motion include at least the dissipation rate per unit mass ε ($\text{m}^2\text{sec}^{-3}$) and the kinematic viscosity ν ($\text{m}^2\text{sec}^{-1}$). With these parameters, the following length, time, and velocity scales can be found:

$$L_s = (\nu^3/\varepsilon)^{1/4}; \quad T_{ts} = (\nu/\varepsilon)^{1/2}; \quad \hat{V}_s = (\nu\varepsilon)^{1/4} \quad (7)$$

2. In the outer region, dominated by large eddies the characteristic scales of the standard k - ε model is used, which are:

$$L_o = (k^{3/2}/\varepsilon); \quad T_{to} = (k/\varepsilon); \quad \hat{V}_o = k^{1/2} \quad (8)$$

The eddy viscosity μ_t can be related to the turbulent kinetic energy k and local time scale as:

$$\mu_t \propto k T_t \quad (9)$$

In order to take into account the effect of the near wall region as well as the outer region, the local time scale must be a function of the Kolmogorov turbulent time scale (7) and the outer region time scale (8) as:

$$T_t = T_{to} + f_T T_{ts} = T_{to} \left(1 + f_T \frac{T_{ts}}{T_{to}} \right) = \frac{k}{\varepsilon} \left(1 + \frac{f_T}{\sqrt{Re_t}} \right) \quad (10)$$

where $Re_t = k^2/(\nu\varepsilon)$ is the turbulent Reynolds number, usually used in the near wall turbulence models, and f_T is a damping function which must satisfy the following asymptotic conditions:

$$f_T \rightarrow 1 \text{ in the near the wall region, or when } Re_t \rightarrow 0 \quad (11a)$$

$$f_T \rightarrow 0 \text{ in the outer region, or when } Re_t \rightarrow \infty \quad (11b)$$

Therefore, to satisfy the above conditions the following function is used:

$$f_T = 1 - \exp(-1/Re_t) \quad (12)$$

Substituting for the local time scale (10) in (9) and introduce the proportionality constant and the damping function, the eddy viscosity becomes:

$$\mu_t = \rho C_\mu f_\mu^* \frac{k^2}{\varepsilon} \left\{ 1 + \frac{1 - \exp(-1/Re_t)}{\sqrt{Re_t}} \right\} \quad (13)$$

where f_μ^* is a damping function. The role of this damping function in the low-Reynolds number k - ε models is similar to the Van-Driest damping function in the mixing length turbulence model. In order that μ_t vanishes near the wall, f_μ^* must approach zero when y^+ goes to zero. Then the damping function f_μ^* must satisfy the near wall analytical limiting behavior of the turbulent quantities, which is according to Patel et al [8]:

$$y \rightarrow 0 \quad \frac{\partial U}{\partial y} \propto y^0; \quad u \propto y^1; \quad v \propto y^2; \quad k \propto y^2; \quad \varepsilon \propto y^0; \quad (14)$$

which leads to

$$\mu_t \propto y^3; \quad Re_t \propto y^4; \quad f_\mu^* \propto y^1; \quad (15)$$

To recover the effect of the logarithmic and outer region, Nagano and Hishida [9] have used the following function:

$$f_\mu^* \propto \left\{ 1 - \exp(-y^+ / A^+) \right\}^2 \quad (16)$$

In order to satisfy the near wall conditions (15) the following function is introduced:

$$f_\mu^* = \left\{ 1 - \exp(-y^+ / A^+) \right\}^2 \left\{ 1 + \frac{B^+}{Re_t^{1/4}} \right\} \quad (17)$$

where $A^+ = 38.2$ and $B^+ = 0.1$ are the constants which gives the best results comparing with the experimental data. Therefore if the classical formula of the eddy viscosity defined in equation (4) is used, then the present model damping function becomes:

$$f_\mu = \left\{ 1 - \exp(-y^+ / A^+) \right\}^2 \left\{ 1 + \frac{B^+}{Re_t^{1/4}} \right\} \left\{ 1 + \frac{1 - \exp(-1/Re_t)}{\sqrt{Re_t}} \right\} \quad (18)$$

This damping function satisfies the near wall conditions as well as in the outer region when $y \rightarrow \infty$; $y^+ \rightarrow \infty$; $Re_t \rightarrow \infty$ which leads to $f_\mu \rightarrow 1$.

Improvement is made also in the dissipation transport equation. When the local time scale defined in equation (10) is used in the model form of the source/sink term in the ε -equation, the asymptotic conditions satisfied. In the present study the dissipation transport equation is given in equation (6) with the time scale improvement (10). In the near wall region, ε -equation reduces to

$$0 = \mu \left(\frac{d^2 \varepsilon}{dy^2} \right) - \left(\frac{C_{\varepsilon 2} f_2 \rho \varepsilon}{T_t} \right) \quad (19)$$

substituting for the local time scale, to find

$$\mu \left(\frac{d^2 \varepsilon}{dy^2} \right) = C_{\varepsilon 2} f_2 \rho \frac{\varepsilon^2}{k} \left(\frac{\sqrt{Re_t} + f_T}{\sqrt{Re_t}} \right) \quad (20)$$

Applying the asymptotic conditions (14) on the above equation, the following relation can be obtained: $f_2 \propto y^2$ when $y \rightarrow 0$. And note that $f_2 \rightarrow 1$ when $y \rightarrow \infty$. Therefore, to satisfy the condition of the both regions, the following function is used

$$f_2 = 1 - \exp \left\{ - \left(y^+ / C^+ \right)^2 \right\} \quad (21)$$

where $C^+ = 0.1$ is based on the computer optimization, while the damping function f_1 is assumed to be unity in the present model. Finally the wall boundary condition for the turbulent dissipation rate is given by:

$$\varepsilon(0) = 2\nu \left(\frac{\partial \sqrt{k}}{\partial y} \right)^2 \quad (22)$$

and the other boundary conditions are as usual $\varepsilon(\infty) = 0$, $k(0) = 0$, and $k(\infty) = 0$. To close the model, the constants appearing in eddy viscosity formulation and the transport equations should be specified. To choose the values of these constants, the standard model constants must be used in order to recover the outer region of turbulent boundary layer. Therefore the following standard model constants are used:

$$C_\mu = 0.09; \quad \sigma_k = 1.00; \quad \sigma_\varepsilon = 1.30; \quad C_{\varepsilon 1} = 1.44; \quad C_{\varepsilon 2} = 1.92 \quad (23)$$

4. RESULTS AND DISCUSSION

The boundary layer equations and the modeled transport equations were discretised and solved using integration over control volumes, coupled with interpolation assumptions, in such a way as to link downstream (unknown values) implicitly, given by Spalding and Patankar [10]. This method uses the Patankar-Spalding coordinate system (χ, ω) , where χ is the direction of the flow, and ω is the dimensionless stream function. A non-uniform grid is used to resolve the sharp gradient near the wall. The grid distribution is controlled by the following formula:

$$\omega_i = \left(\frac{i-1}{N-1} \right)^2, \quad i = 1, 2, \dots, N \quad (24)$$

where N is the total number of grid points, in the present work 200 grid points were used to get fine grid near the wall. The marching step size δx is also varied to ensure accuracy.

In order to validate the performance of the present model, one DNS data and two different experimental data are used for comparison. The first example is for the direct numerical simulation DNS data carried out by Spalart [11]. Figure 1 shows the comparison of the dimensionless velocity profile, Reynolds stress, turbulent kinetic energy, and dissipation rate at a location where the Reynolds number based on the momentum thickness $Re_\theta = 1410$.

As shown in figure 1, the maximum discrepancy between the DNS data and the present prediction is in the dissipation rate profile especially at and near the wall regions. This may be due to the choice of the boundary condition for the ε -equation as given by equation (22), which is based on the polynomial expansion of the dimensionless kinetic energy k^+ in terms of y^+ near the wall. Further research on the modeling of the ε -equation and its boundary condition at the wall is needed to improve the results.

The next comparison is chosen for high Reynolds number flow, which is the experimental measurement of Wieghardt IDENT 1400 from the 1968 AFOSR-IFP-Stanford Conference [12]. The model was a waxed-plywood plate with a slightly undulating surface, the tripping device was a blunt leading edge fitted with a small trip wire. The tunnel turbulence level was about 0.25 %. The first profile is probably at about the minimum Reynolds number of turbulent flow. The measured values of the mean velocity profile at the last measured station ($\chi = 4.987 \text{ m}$) are presented in figure 2 together with the numerical results obtained from the present model. Also the comparison of the skin friction along the boundary layer of the same flow is presented in figure 3.

In this case also a good agreement between the experimental data and the present prediction method is observed.

(a) Mean velocity profile

(b) Reynolds stress profile

(c) Turbulent kinetic energy profile

(d) Dissipation rate profile

Figure 1. Comparison with DNS [11] of dimensionless mean velocity, Reynolds stress, turbulent kinetic energy, and dissipation rate at $Re\theta = 1410$.
($\circ$ DNS; — Present).

Finally, the effect of Reynolds number on turbulent boundary layer turbulence is studied recently by DeGraaff et. al. [13]. The test section used is 2.9 m long, with a 150 by 710 mm rectangular test section. The nominal operating velocity is 15 m/s, with a free stream turbulence level of 0.2%.

The experimental results were presented for the zero pressure gradient flat plate boundary layer over a momentum thickness Reynolds number range 1500-15000. A comparison of the mean velocity profiles is carried out with the minimum and maximum Reynolds

Figure 2. Comparison of the dimensionless mean velocity profile.
(o Wieghard [12] — Present).

Figure 3. Comparison of the skin friction distribution.
(o Wieghard [12] — Present).

number as shown in figure 4 using the inner coordinates. And figure 5 shows a comparison of the same data plotted in the outer coordinate. In this case encouraging results are also obtained and the good agreement is clear between the experimental data and the present prediction method.

Figure 4. Comparison of the dimensionless mean velocity profile, inner coordinate (o DeGraaff et. al.[13] at $Re\theta = 1500$, + DeGraaff et. al.[13] at $Re\theta = 15000$; — Present at $Re\theta = 1500$, - - - Present at $Re\theta = 15000$).

Figure 5. Comparison of the dimensionless mean velocity profile, outer coordinate (o DeGraaff et.al.[13] at $Re\theta = 1500$, + DeGraaff et.al.[13] at $Re\theta = 15000$; — Present at $Re\theta = 1500$, - - - Present at $Re\theta = 15000$).

5. CONCLUSIONS

In this paper, a new method of modeling the eddy viscosity and the dissipation transport equation is used for the turbulent boundary layer flows as an extension of the k - ϵ model of turbulence. This method is based on dividing the turbulent boundary layer into two

regions: near wall and outer regions, and defining the local turbulent time scale for both region and combined them by a time damping function. This turbulent time scale is used to model the eddy viscosity and the source/sink term in the dissipation rate equation. The present model used the same set of the empirical constants of the standard k - ε model to ensure accuracy in the high Reynolds number flows.

The prediction results show a good agreement in prediction turbulent boundary layer flow in the low as well as high Reynolds number flows.

ACKNOWLEDGEMENT

The author wish to thank the Universiti Sains Malaysia for providing facilities and funding under a short-term grant No. 073466 to carry out the research.

NOTATION

A^+, B^+, C^+	turbulence model Constants	u, v	fluctuating velocity components
$C_\mu, C_{\varepsilon 1}, C_{\varepsilon 2}$	turbulence model Constants	$-\overline{uv}^+$	normalized Reynolds stress, $-\overline{uv}^+ = -\overline{uv}^+ / u_\tau^2$
C_f	skin friction coefficient $C_f = \frac{2\tau_w}{\rho U_\infty^2}$	u_τ	friction velocity, $u_\tau = \sqrt{\tau_w / \rho}$
f_I, f_2, f_T, f_μ	turbulence model damping functions	x, y	cartesian coordinates
k	turbulent kinetic energy	$y^\mp$	normalized $y, y^+ = \frac{yu_\tau}{\nu}$
k^+	normalized $k, k^+ = k / u_\tau^2$	ε	dissipation rate of turbulent kinetic energy
L	length scale of turbulence	ε^+	normalized $\varepsilon, \varepsilon^+ = \frac{\nu \varepsilon}{u_\tau^4}$
N	total number of crossstream nodes	θ	boundary layer momentum thickness
P	mean pressure	ν	kinematic viscosity
Re_t	turbulent Reynolds number	μ	dynamic viscosity
Re_θ	momentum thickness Reynolds number	μ_t	turbulent viscosity
T_t	time scale of turbulence	ρ	fluid density
U, V	mean velocity components	$\sigma_k, \sigma_\varepsilon$	turbulence model constants
U_∞	free stream velocity	τ_w	wall shear stress
U^+	normalized $U, U^+ = \frac{U}{u_\tau}$	ω	dimensionless stream function
$\hat{V}$	velocity scale of turbulence		

REFERENCES

1. Launder, B.E. and Spalding, D.B., The numerical computation of turbulent flow, *Comp. Meth. in Appl. Mech. and Eng.*, 1974,3, pp. 269-289
2. Jones, W.P. and Launder, B.E., The prediction of laminarization with two-equation model of turbulence. *Int. J. Heat Mass Transfer*, 1972, 15, 301-314.
3. Bradshaw, P., Launder, B.E. and Lumley, J.L., Collaborative testing of turbulence models. *ASME J. of Fluids Engineering*, 1996, 118, pp. 243-247.
4. Rodi, W., Turbulence models for environmental problems. Prediction methods for turbulent flows, McGraw-Hill, New York, 1980.
5. Lai, J.C.S. and Yang, C.Y., Numerical simulation of turbulence suppression: Comparisons of the performance of four k - ϵ turbulence models. *Int. J. Heat and Fluid Flow*, 1997, 18, pp. 575-584.
6. Sarkar, A. and So, R.M.C., A critical evaluation of near-wall two-equation models against direct numerical simulation data. *Int. J. Heat and Fluid Flow*, 1997, 18, pp. 197-208.
7. Tennekes, H. and Lumley, J.L., A first course in turbulence, The MIT Press. Cambridge, 1990.
8. Patel, V.C., Rodi, W. and Scheuerer, Turbulence models for near-wall and low Reynolds number flows: A review, *AIAA J.*, 1985, 1, pp. 33-38.
9. Nagano, Y. and Hishida, M., Improved form of the k - ϵ model for wall turbulent shear flows, *ASME J. of Fluids Engineering*, 1987, 109, pp. 156-160.
10. Spalding, D.B., GENMIX - a general computer program for two dimensional parabolic phenomena, Pergamon, Oxford, 1977.
11. Spalart, P.R., Direct simulation of a turbulent boundary layer up to $Re_\theta = 1410$, *J. of Fluid Mech.*, 1988, 187, pp. 61-98.
12. Coles, D.E. and Hirst, E.Å., Computation of turbulent boundary layer, 1968 AFOSR-IFP-Stanford Conference, Stanford University, Palo Alto, CA, 1969.
13. DeGraaff, D.B., Webster, D.R. and Eaton, J.K., The effect of Reynolds number on boundary layer turbulence, *Experimental Thermal and Fluid Science*, 1999, 18, pp. 341-346.